Coulomb gap in one-dimensional disordered electron systems

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The density of states of one-dimensional disordered electron systems with long range Coulomb interaction is studied in the weak pinning limit. The density of states is found to follow a power law with an exponent determined by localization length, and this power law behavior is consistent with the existing numerical results.

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Introduction-Recently there have been much interests in the one-dimensional (1D) electron systems motivated by the development of carbon nanotube technology.[1] In 1D electron systems electron-electron interactions play very important roles, leading to phases different from the conventional Fermi liquids.[2] Repulsive short range electron-electron interactions cause Luttinger liquids (LL),[2] while long range Coulomb interaction (LRCI) is believed to cause a Wigner crystal.[3] The Wigner crystal phase of spinful electrons is characterized by quasi long range order of $4k_F$ charge density components.

$$\langle \rho_{4k_F}(x)\rho_{4k_F}(0)\rangle \sim e^{-\sqrt{\ln x}}. \tag{1}$$

Eq. (1) should be compared with the power law dependence $x^{-\alpha}$ of LL. Eq. (1) also indicates the suppressed quantum fluctuations of charge densities, and this feature manifest in the density of states (DOS) $D(\omega)$.[3]

$$D(\omega) \sim \exp\Big[- \text{const.} \left(\ln \frac{E_c}{|\omega|}\right)^{3/2}\Big], \tag{2}$$

where ω is measured from Fermi energy, and E_c is a cut-off energy. Note that DOS of Eq. (2) decays faster than any other power law. For LL, DOS follows a power law $|\omega|^\gamma$ with a nonuniversal positive γ. The physics of 1D Wigner crystal is analogous to that of charge density wave (CDW)[4,5,6] apart from the presence of quantum fluctuations and LRCI.

Impurities, either a few or many, change the physical properties of 1D electron systems qualitatively. For a single impurity in an electron system with repulsive interaction, the back scattering of electrons with the impurity becomes strong at low energy, and it effectively divides the system into two pieces.[7,8]

For noninteracting electrons in random disorder (many impurities), all the states are known to be localized due to repeated back scattering.[9] DOS for a disorder with a Gaussian distribution (with zero mean) can be calculated exactly.[10,11]

$$\begin{aligned} D(E) &= \frac{1}{\pi\sqrt{2E}}, \quad \text{for} \quad E \to \infty \\ D(E) &= \frac{8(-E)}{3\pi}\, e^{-\frac{1}{12}(-8E)^{3/2}}, \quad \text{for} \quad E \to -\infty. \end{aligned} \tag{3}$$

In the strongly localized electron systems with LRCI, where the overlap of wave functions can be neglected, electrons can be treated classically and DOS exhibits a Coulomb gap of the form:[12]

$$D(\omega) \sim \left(\ln \frac{E_c}{|\omega|}\right)^{-1}. \tag{4}$$

We note that Eq. (4) has been derived under the assumption that the localization length is much smaller than the interparticle distance. Both disorder and LRCI push a 1D electron system to classical limit but in a different manner as is reflected in the form of DOS Eq. (2) and Eq. (4). In this paper we report a result on the DOS at low energy for the 1D disordered electron system with LRCI when the localization length is larger than the interparticle distance or interimpurity distance.[more precisely the weak pinning limit, see below] Following the analyses on a pinned Wigner crystal by Maurey and Giamarchi[13,14], employing a simplified model, and using semiclassical approximation we find that DOS follows a *power law* at low energy.

$$D(\omega) \sim |\omega|^{\sqrt{1+\eta}/2}, \tag{5}$$

where the exponent η is basically determined by the localization length.[see Eq. (15)] This power law behavior is consistent with the existing numerical results.[15] Eq. (5) is the main result of this paper.

Model- We consider a spinless electron system for simplicity. Such a system can be realized in organic chains and quasi-1D quantum wires in strong magnetic field.[16] The Hamiltonian consists of three parts.

$$H = H_0 + H_{\text{coul}} + H_{\text{imp}},$$

$$H_0 = v_F \int dx \Big[- i\psi_R^\dagger \partial_x \psi_R + i\psi_L^\dagger \partial_x \psi_L \Big], \tag{6}$$

$$H_{\text{coul}} = \int dxdy \frac{V(x-y)}{2} \rho(x)\rho(y). \tag{7}$$

The operator $\psi_R(\psi_L)$ is the right-moving (left-moving) electron operator. The continuum chiral electrons and lattice electron operators are related by

$$c(x) = \sqrt{a}\Big[e^{ik_F x}\psi_R(x) + e^{-ik_F x}\psi_L(x)\Big], \tag{8}$$

where a is lattice constant.

$\rho_R =: \psi_R^\dagger \psi_R :$ is the normal ordered right-moving edge electron density operator (ρ_L is similarly defined), and $\rho(x) = \rho_R(x) + \rho_L(x)$. $V(x) = \frac{e^2}{\epsilon}\frac{1}{\sqrt{x^2+d^2}}$ is the Coulomb interaction. d is the transverse size of quantum wire which we take to be the same a for simplicity. ϵ is a dielectric constant. The Coulomb matrix element is $V(k) = \frac{2e^2}{\epsilon}K_0(a|k|) \sim \frac{2e^2}{\epsilon}\ln\frac{1}{|k|a}$ for $|k|a > 1$ and K_0 is the modified Bessel function. The impurity Hamiltonian is given by

$$H_{\rm imp} = \sum_x W_I(x) c^\dagger(x) c(x)$$
$$= \int dx W_I(x)\Big[\rho(x) + e^{2ik_F x}\psi_L^\dagger(x)\psi_R(x) + \text{H.c}\Big], \quad (9)$$

where $W_I(x)$ is the impurity potential. The first term in the bracket of Eq. (9) is the forward scattering term which can be neglected compared to back scattering at low energy. The impurity potential $W_I(x)$ is chosen to be

$$W_I(x) = \sum_j V_0 \delta(x - X_j), \quad (10)$$

where X_j's are the random locations of impurities. The interacting electron systems can be bosonized in a standard way.[2,17] The phase fields and bosonization formulas are given by

$$\rho_R + \rho_L = \frac{1}{\pi}\partial_x\theta, \quad \rho_R - \rho_L = \frac{1}{\pi}\partial_x\phi,$$
$$\psi_R = \frac{e^{i\theta+i\phi}}{\sqrt{2\pi a}}, \quad \psi_L = \frac{e^{-i\theta+i\phi}}{\sqrt{2\pi a}}. \quad (11)$$

The bosonized Hamiltonian of the system is

$$H = \int dx \frac{v_F}{2\pi}\Big[(\partial_x\theta)^2 + (\partial_x\phi)^2\Big]$$
$$+ \frac{1}{2\pi^2}\int dxdy\Big[V(x-y)\,\partial_x\theta(x)\,\partial_y\theta(y)\Big]$$
$$+ \sum_j V_0\rho_0\cos[2k_F X_j + 2\theta(X_j)], \quad (12)$$

where ρ_0 is the average density of electrons.[13]

Pinning Length-The pinning length L_0 is a length scale over which the phase field $\theta(x)$ in the ground state varies with $\delta\theta \sim \pi$ in order to take advantage of the impurity potential.[13,14] The pinning length corresponds to the localization length of the electron system. The pinning length can be obtained by maximizing the energy gains from impurity, elastic, and Coulomb energies.[5,13,14] Including the effect of quantum fluctuations using the self-consistent harmonic approximation, the pinning length is given by

$$L_0 = \left(\frac{8e^2}{\pi^2 V_0\rho_0\gamma n_i^{\frac{1}{2}}}\right)^{\frac{2}{3}} \ln^{\frac{2}{3}}\left[\frac{1}{a}\left(\frac{8e^2}{\pi^2 V_0\rho_0\gamma n_i^{\frac{1}{2}}}\right)^{\frac{2}{3}}\right], \quad (13)$$

where γ is a numerical constant characterizing short-range interaction and quantum fluctuations and n_i is impurity concentration. The logarithmic factor is due to LRCI, which enhances the pinning length. The enhanced pinning length implies the more rigid system and the more difficult pinning by impurties. The system becomes much more ordered and the fluctuations around the ground state is much less important. The expression for pinning length Eq. (13) has been derived under the assumption of *weak pinning* $L_0 \gg n_i^{-1}$.

Beyond the pinning length scale the phase coherence of θ is lost, thus the correlation function of CDW operator is expected to decay exponentially[6] ($|x - x'| > L_0$)

$$\langle\cos[2\theta(x,\tau)]\,\cos[2\theta(x',\tau)]\rangle \sim e^{-|x-x'|/L_0}. \quad (14)$$

Eq. (14) implies that the system breaks into segments whose typical length is given by the pinning length L_0. Thus at low energy we can consider a typical segment with length L_0 and calculate DOS averaged over the segment. Since the tunneling between segments is strongly suppressed at low energy we have to fix the value of θ at the boundary of a particular segment.[7](Dirichlet boundary condition)

Approximate Low Energy Model- Let us construct a model for a segment which is valid at low energy. First of all the logarithmic divergence of Coulomb matrix element $V(k)$ is cut by $k \sim 1/L_0$. Then the Coulomb energy term can be expressed as

$$H_{\rm coul} \sim \eta\frac{v_F}{2\pi}\int_0^{L_0} dx[\partial_x\theta(x)]^2,$$
$$\eta = \frac{2e^2}{v_F\pi\epsilon}\ln\frac{L_0}{a}. \quad (15)$$

The effective Hamiltonian of a segment becomes

$$H_{\rm seg} = \int_0^{L_0} dx\frac{v_F}{2\pi}\Big[(\partial_x\theta)^2(1+\eta) + (\partial_x\phi)^2\Big]$$
$$+ V(\theta, X_j), \quad (16)$$

where $V(\theta, 0 < X_j < L_0)$ is some potential energy term which is optimized by a certain random *average* phase value $\theta_{\rm opt}$ and it depends on θ only through $e^{\pm 2i\theta}$.[see the last line of Eq. (12)] Note that $\theta \to \theta + \pi$ is then a symmetry of the system. In the weak pinning limit we are considering the phase θ varies rather smoothly in the range $|\theta - \theta_{\rm opt}| \le \pi$ at low energy over a length scale L_0.

Calculation of DOS- The electron Green function in imaginary time is defined by

$$G_{R/L}(x, y, \tau_1 - \tau_2, W_I) = -\langle T_\tau\,\psi_{R/L}(x,\tau_1)\psi_{R/L}^\dagger(y,\tau_2)\rangle \quad (17)$$

for a particular realization of impurities W_I. After averaging ove impurities

$$\bar{G}_{R/L}(x-y, \tau_1-\tau_2) = \langle G_{R/L}(x, y, \tau_1-\tau_2, W_I)\rangle_{W_I}. \quad (18)$$

DOS is defined by

$$D(\omega) = -\frac{1}{\pi}\,\mathrm{Im}\Big[\int_0^\beta d\tau e^{i\omega\tau}\bar{G}_{R/L}(0,\tau)\Big]_{i\omega\to\omega+i\delta}. \quad (19)$$

The above expressions are formally exact but difficult to calculate. Here instead we will calculate the electron Green function at a point y in the segment (away from the boundary) using the Hamiltonian Eq. (16).

$$\langle G_{R/L}(y,y,\tau_1-\tau_2)\rangle_{W_I}. \quad (20)$$

For the computation of electron Green function, the Lagrangian formulation is more convenient.

$$S_{\rm seg} = \int_0^{L_0} dx \int_0^\beta d\tau \frac{i}{\pi}\partial_\tau\theta\,\partial_x\phi + \int_0^\beta d\tau H_{\rm seg}. \quad (21)$$

Using the bosonization formula $G_{R/L}(y,\tau_1-\tau_2)$ can be expressed as

$$\begin{aligned} G_R(y,\tau_1-\tau_2) &= \frac{\int D[\theta,\phi]\, e^{-S_{\rm seg}-S{\rm ex}}}{\int D[\theta,\phi] e^{-S_{\rm seg}}},\\ S_{\rm ex} &= i\int dx d\tau\Big[\theta(x,\tau)+\phi(x,\tau)\Big]J(x,\tau),\\ J(x,\tau) &= \delta(x-y)\Big[\delta(\tau-\tau_1)-\delta(\tau-\tau_2)\Big]. \end{aligned} \quad (22)$$

In Eq. (22) the dual phase field ϕ can be integrated out explicitly. [Remember that the potential term $V(\theta,X_j)$ depends only on θ not on ϕ.] The electron Green function becomes

$$G_R(y,\tau_1-\tau_2) = \frac{\int D[\theta]\, e^{-S_\theta - S_{J\theta}-S_J}}{\int D[\theta]\, e^{-S_\theta}}. \quad (23)$$

$$\begin{aligned} S_\theta &= \frac{u}{2\pi K}\int_0^{L_0} dx\int_0^\beta d\tau\left[\frac{1}{u^2}(\partial_\tau\theta)^2+(\partial_x\theta)^2\right]\\ &+ \int_0^\beta d\tau\, V(\theta,X_j),\\ K &= \frac{1}{\sqrt{1+\eta}},\quad u = v_F\sqrt{1+\eta}. \end{aligned} \quad (24)$$

$$\begin{aligned} S_{J\theta} &= -\frac{1}{2K}\sum_{\omega,k}\frac{\omega}{uk}\Big[J(-i\omega,-k)\theta(i\omega,k)+\text{H.c.}\Big]\\ &+ i\int dx d\tau\,\theta(x,\tau)\,J(x,\tau). \end{aligned} \quad (25)$$

$$S_J = \frac{\pi}{2K}\sum_{\omega,k}\frac{J(-i\omega,-k)J(i\omega,k)}{uk^2}. \quad (26)$$

Because our system is in the deep classical regime the dominant low energy processes would come from the quantum tunneling between classical vacua. Remembering that the phase field θ is an angular variable, the classical vacua are charerized by $\theta_{\rm vac}\sim\{\theta_{\rm opt}+\pi n, n=\text{integer}\}$. This is because the potential $V(\theta,X_j)$ depends on θ only through the form $e^{\pm 2i\theta}$, so that it possesses the symmetry

$$\theta\to\theta+\pi \quad (27)$$

Since the phase field θ varies on the order π in each segment one can expect that the most domiant process would be the quantum tunneling between $\theta_{\rm opt}$ and $\theta_{\rm opt}\pm\pi$ vacua. The quantum tunneling processes between classical vacua can be described by the solutions of classical equations of motion in *imaginary time*, which are also called *instantons*.[18]

Now we have to find the classical *(imaginary) time dependent* solution which minimizes $S_\theta+S_{J\theta}$. Since $S_{J\theta}$ is linear in θ it plays a role of the external source field. We argue below that the first term of $S_{J\theta}$ [Eq. (25)] is the source for the vortex and anti-vortex configuration of the (angular) phase field θ. For the moment let us neglect the potential term $V(\theta,X_j)$. Then the phase field θ and its dual phase field ϕ are related by the Cauchy-Riemann equation[19] [we set $u=1$ for simplicity from now on]

$$\partial_\mu\phi = -i\epsilon_{\mu\nu}\partial_\nu\theta,\ \ \mu,\nu=\tau,x, \quad (28)$$

where $\epsilon_{\mu\nu}$ is a totally antisymmetric tensor. The first term Eq. (25) stems from the source of dual field ϕ.[see Eq. (22)] Including only one source at (τ_1,y) the ϕ field satisfies the classical equation of motion.

$$-(\partial_\tau^2+\partial_x^2)\phi(x,\tau) = i\pi\delta(x-y)\delta(\tau-\tau_1). \quad (29)$$

Combining Eq. (28) and Eq. (29) we deduce

$$\pi\delta(x-y)\delta(\tau-\tau_1) = \epsilon_{\mu\nu}\partial_\mu\partial_\nu\theta(x,\tau), \quad (30)$$

Naively the right hand side of Eq. (30) vanishes but this is not necessarily true for the topologically singular configuration such as vortex. The vortex nature of θ is demonstrated by the calculation of vorticity using Eq. (30) and Stokes theorem.

$$\oint dx_\mu\,\partial_\mu\theta = \int d^2x\epsilon_{\mu\nu}\,\partial_\mu\partial_\nu\theta = \frac{1}{2}2\pi, \quad (31)$$

which characterizes the vortex nature of θ.[20] The unusual factor of 1/2 is due to the aforementioned symmetry $\theta\to\theta+\pi$. Remembering that we have another souce at (τ_2,y) with a charge opposite to that at (τ_1,y)[see the last line of Eq. (22)], we conclude that the vortex-antivortex configuration of θ phase field will dominate the electron Green function at long time in the classical regime. Before proceeding to the explicit vortex-antivortex instanton solution, let us discuss another classical field configurations generated by the second source term of Eq. (25). It satisfies the standard Laplace equation in two dimension with delta function sources, and its explicit form is given by

$$\theta_0(\tau,x)\sim\frac{K}{4}\ln\frac{(x-y)^2+(\tau-\tau_1)^2}{(x-y)^2+(\tau-\tau_2)^2}. \quad (32)$$

In the configuration of Eq. (32) the range of variation of phase field θ is much large than π for long time limit $|\tau_1 - \tau_2| \to \infty$. Thus the configuration $\theta_0(x)$ is suppressed by the potential term $V(\theta, X_j)$, and it also does not exhibit the angular nature of phase fields. This is an analogue of the spin-wave degrees of freedom of XY model.[18] Thus, the configuration of Eq. (32) does not contribute the electron Green function in long time limit.

The explicit vortex-antivortex solution in complex coordinate is given by[18]

$$e^{i2\theta(z)} = \frac{z - (\tau_1 + iy)}{z - (\tau_2 + iy)}, \;\; z = \tau + ix \tag{33}$$

Substituting Eq. (33) into the action S_θ we obtain

$$\begin{aligned} G_R(y, \tau_1 - \tau_2) \;&\sim\; \exp[-\frac{1}{2K} \ln \frac{|\tau_1 - \tau_2|}{\tau_0}\Big], \\ &= \;\Big[\frac{|\tau_1 - \tau_2|}{\tau_0}\Big]^{-1/2K}, \end{aligned} \tag{34}$$

where τ_0 is the short time cut-off provided by the contributions from the vortex cores. Eq. (34) implies Eq. (5).

To complete our analysis let us compute S_J Eq (26). The divergent k summation at small k should be cut by $1/L_0$. The frequency summation diverges at large frequency (short time) which is cut by some high energy scale such as $v_F n_i$.[8] Thus, S_J does not contribute to the electron Green function in long time limit.

According to Eq. (5) the exponent of DOS is larger for the longer localization length, which is consistent with the numerical results by Jeon *et al.*.[15] The exponent η can be expressed as

$$\eta = \frac{1}{137} \times \frac{2c}{v_F} \times \frac{1}{\epsilon} \times \ln \frac{L_0}{a}, \tag{35}$$

where c is the speed of light. In the weak pinning limit it is reasonable to take $\ln \frac{L_0}{a}$ to be 4-6. Choosing a typical Fermi velocity near 10^7cm/sec and $\epsilon \sim 1 - 5$ for a quantum wire, we can estimate the exponent of DOS $\sqrt{1+\eta}/2$ to be around 3 - 6 which is also consistent with the exponents obtained from the numerical studies.[15]

More precisely the Eq. (34) should be averaged over the impurity configurations in the *segment*. Inside the segment the phase field varies smoothly in space and the impurity average is not expected to bring in any singular effects which would invalidate Eq. (34), since no infrared divergence can occur through impurity averages for a finite segment.

Sufficiently strong short range interaction can also push the system to classical regime ($K << 1$), where the result Eq. (5) is applicable. On the contrary, the result Eq. (5) is not applicable to the case of noninteracting disordered electrons ($K = 1$) since the condition of the classical limit ($K << 1$) is not met. Indeed the DOS of noninteracting disordered electrons is finite at the Fermi level (See Eq. (3)) contrary to Eq. (5).

Summary- We have studied analytically the DOS of the disordered 1D electron system interacting via long range Coulomb interaction employing a simplified model and semiclassical approximation. The DOS is found to follow a power law with a nonuniversal exponent which is basically determined by localization length. The power law dependence is also consistent with the existing numerical result.

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